

Alternative parametrizations and reference priors for decomposable discrete graphical models

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Abstract

For a given discrete decomposable graphical model, we identify several alternative parametrizations, and construct the corresponding reference priors for suitable groupings of the parameters. Specifically, assuming that the cliques of the graph are arranged in a perfect order, the parameters we consider are conditional probabilities of clique-residuals given separators, as well as generalized log-odds-ratios. We also consider a parametrization associated to a collection of variables representing a cut for the statistical model. The reference priors we obtain do not depend on the order of the groupings, belong to a conjugate family, and are proper.

Some key words: Clique; Conjugate family; Contingency table; Cut; Log-linear model; Multinomial model; Natural exponential family.

1 Introduction

Graphical models, see e.g. Lauritzen (1996), are statistical models such that dependencies between variables are expressed by means of a graph. The study of graphical models is an established and active area of applied and theoretical research. Directed graphs for discrete variables, often called Bayesian networks, see e.g. Cowell et al. (1999), have been used in a variety of applied domains, and represent the engine of probabilistic expert systems. On the other hand, undirected graphical models for the analysis of discrete data are best employed for the analysis of multi-way contingency tables, and represent a useful subset of hierarchical log-linear models .

In this paper we are concerned with the Bayesian analysis of discrete undirected graphical models, whose underlying graph is decomposable. When working in a Bayesian framework, a prior distribution on the parameter space is required. Priors for undirected discrete graphical, or more generally, log-linear models have been considered in Dawid and Lauritzen (1993), Madigan and York (1995), Dellaportas and Forster (1999), Kings and Brooks (2001), Dellaportas and Tarantola (2005).

Despite the adoption of reasonably simplified models, prior elicitation still represents a major concern even for moderately large graphs, because of the very high number of parameters involved. This naturally suggests to search for default, or objective, priors, requiring a minimal subjective input and essentially model-based. However there is now evidence, see e.g. Berger (2000) and Casella (1996), that naive approaches based on flat non-informative priors are largely inadequate in multi-parameter settings. In this context, reference analysis provides one of the most successful general methods to derive default prior distributions. For a recent and informative review see Bernardo (2005). While the algorithmic complexity for the construction of reference priors can be substantial, it is known that suitable re-parametrizations of the

model may considerably simplify the task, see for instance Consonni et al. (2004) and Consonni and Leucari (2006).

We address two specific issues in this paper: identifying alternative parametrizations for a given discrete graphical model, and constructing the corresponding reference priors. More precisely, in §2 we consider several parametrizations: conditional probabilities of clique-residuals given separators, as well as generalized log-odds ratios that arise as canonical parameters of equivalent exponential family representations of the underlying sampling distribution, and explicate their mutual relationships. In §3 we provide the expressions for the corresponding reference priors, and discuss their main properties. In §4 we present a parametrization associated to a cut in the graphical model and derive the corresponding reference prior. Some points for discussion are summarized in the last section. Technical details for the proof of the relationships between various parametrizations are given in the Appendix.

2 Generalized log-odds-ratios parametrizations

2.1 Preliminaries

Let us recall some basic facts about undirected graphs and graphical models: for further details the reader is referred to Lauritzen (1996, ch. 2). An undirected graph G is a pair (V, E) where V is a finite set of vertices and E the set of edges, an edge being an unordered pair $\{\gamma, \delta\}$, $\gamma \in V, \delta \in V, \gamma \neq \delta$. Henceforth the graph G is assumed to be decomposable. For a given ordering C_1, \dots, C_k of the cliques, we will use the following notation

$$H_l = \cup_{j=1}^l C_j, l = 1, \dots, k, \quad S_l = H_{l-1} \cap C_l, \quad l = 2, \dots, k, \quad R_l = C_l \setminus S_l, \quad l = 2, \dots, k.$$

A given ordering of the cliques is said to be perfect if for any $l > 1$ there is an $i < l$ such that $S_l \subseteq C_i$. When we have a perfect ordering of the cliques, the $S_l, l = 2, \dots, k$ are minimal separators. The H_l and R_l are called respectively, the l -th history and l -th residual.

A graphical model, Markov with respect to a given graph G , is a family of probability distributions on $(X_\gamma, \gamma \in V)$ such that X_δ is independent of X_γ given $X_{V \setminus \{\delta, \gamma\}}$ whenever $\{\gamma, \delta\}$ is not in E .

In this paper we shall focus on contingency tables arising from the classification of N units according to a finite set V of criteria, see Lauritzen (1996, Ch. 4). Each criterion is represented by a variable X_γ , $\gamma \in V$, which takes values in a finite set \mathcal{I}_γ . Let $\mathcal{I} = \times_{\gamma \in V} \mathcal{I}_\gamma$. The cells of the table are the elements

$$i = (i_\gamma, \gamma \in V), i \in \mathcal{I}. \quad (2.1)$$

Each of N individuals falls into cell i independently with a probability $p(i)$; we let $p = (p(i), i \in \mathcal{I})$, with $\sum_{i \in \mathcal{I}} p(i) = 1$. Furthermore, we write $n(i)$ for the i -th cell-count and $n = (n(i), i \in \mathcal{I})$, with $\sum_{i \in \mathcal{I}} n(i) = N$.

We consider here the model \mathcal{M}_G , which, for a given G and a given integer N , is the set of multinomial $\mathcal{M}(N, p)$ distributions with $N = \sum_{i \in \mathcal{I}} n(i)$ and $p = (p(i), i \in \mathcal{I})$ in the $|\mathcal{I}| - 1$ dimensional simplex, which are Markov with respect to G .

From now on, we adopt the notation “ $D \subseteq_0 V$ ” to mean that D may be the empty set while “ $D \subseteq V$ ” excludes the empty set. Let \mathcal{E} denote the power set of V , excluding the empty set, i.e.

$$\mathcal{E} = \{F \subseteq V, F \neq \emptyset\}.$$

For $D \in \mathcal{E}$,

$$i_D = (i_\gamma, \gamma \in D), \text{ and } n(i_D), i_D \in \mathcal{I}_D = \times_{\gamma \in D} \mathcal{I}_\gamma \quad (2.2)$$

denotes a cell in the D -marginal table, and its corresponding count. We therefore have

$$n(i_D) = \sum_{j \in \mathcal{I} | j_D = i_D} n(j) = \sum_{j_{V \setminus D} \in \mathcal{I}_{V \setminus D}} n(i_D, j_{V \setminus D}). \quad (2.3)$$

Note that $n(i_\emptyset) = N$. For F, D in \mathcal{E} , we use the notation $p^D(i_D)$ and $p^{D|i_F}(i_D)$ to denote, respectively, the marginal and the conditional probabilities

$$p^D(i_D) = \sum_{j \in \mathcal{I} | j_D = i_D} p(j) \quad (2.4)$$

$$p^{D|i_F}(i_D) = \frac{p^{D \cup F}(i_D, i_F)}{p^F(i_F)}. \quad (2.5)$$

Assuming that “0” indicates one of the levels for each variable, we let i_γ^* denote the “0”-level in \mathcal{I}_γ , so that

$$i^* = (i_\gamma^*, \gamma \in V)$$

denotes the cell with all components equal to 0.

Definition 2.1 For $D \in \mathcal{E}$, we define

$$\mathcal{I}_D^* = \{i_D \mid i_\gamma \neq i_\gamma^*, \forall \gamma \in D\}. \quad (2.6)$$

In words, \mathcal{I}_D^* is the set of marginal cells i_D such that none of their components is equal to 0. We set $\mathcal{I}_V^* = \mathcal{I}^*$. For example, if $D = \{a, b, c\}$, a takes the values $\{0, 1, 2, 3\}$, b takes the values $\{0, 1, 2\}$, c takes the values $\{0, 1\}$, then

$$\mathcal{I}_D^* = \{(1, 1, 1), (2, 1, 1), (3, 1, 1), (1, 2, 1), (2, 2, 1), (3, 2, 1)\}$$

2.2 The saturated case

We assume here that G is complete and \mathcal{M}_G is therefore the saturated multinomial model for $n = (n(i), i \in \mathcal{I})$. The multinomial probability function is usually written in terms of the cell probabilities $p = (p(i), i \in \mathcal{I})$ as

$$f(n|p) = \frac{N!}{\prod_{i \in \mathcal{I}} n(i)!} \prod_{i \in \mathcal{I}} p(i)^{n(i)}, \quad (2.7)$$

where the only restriction on the parameters $p(i)$ is $\sum_i p(i) = 1$. It is convenient to regard the multinomial coefficient in (2.7) as being part of the dominating measure, so that the actual density is simply $\prod_{i \in \mathcal{I}} p(i)^{n(i)}$. Assuming that all probabilities are positive, the density (2.7), with respect to a suitable dominating measure, can be represented in exponential family form as

$$\prod_{i \in \mathcal{I}} p(i)^{n(i)} = \exp \left\{ \sum_{i \in \mathcal{I}, i \neq i^*} n(i) \xi(i) - N \log \left(1 + \sum_{i \in \mathcal{I}, i \neq i^*} e^{\xi(i)} \right) \right\}, \quad (2.8)$$

where

$$\xi(i) = \log \frac{p(i)}{p(i^*)}, \quad i \in \mathcal{I}^* \quad (2.9)$$

are the usual log-odds, relative to the benchmark cell i^* . We recognize in (2.8) a natural exponential family (henceforth abbreviated NEF), with canonical parameters $\xi(i)$ and canonical statistics $n(i)$, $i \neq i^*$. For a review of NEFs, see e.g. Kotz, Balakrishnan, and Johnson (2000, ch. 54).

In this paper, we shall work with NEF-representations alternative to (2.8), featuring different canonical statistics and their corresponding canonical parametrizations, the latter representing various generalized log-odds-ratios of joint probabilities, residual-conditional probabilities or clique-marginal probabilities. For the saturated model, we need consider only the generalized log-odds-ratio of model probabilities defined as follows.

Definition 2.2 *For all $D \subseteq_0 V$ and $i_D \in \mathcal{I}_D^*$ we define the log-linear parameters*

$$\theta(i_D) = \log \prod_{F \subseteq_0 D} p(i_F, i_{V \setminus F}^*)^{(-1)^{|D \setminus F|}}, \quad (2.10)$$

Note that for $F = \emptyset$, $p(i_F, i_{V \setminus F}^*) = p(i^*)$ and $\theta(i_\emptyset) = \theta(i^*) = \log p(i^*)$. The parameters $\theta(i^*)$ and $p(i^*)$ are not free but functions of the other θ or p parameters. We also emphasize the fact that while $\theta(i_D)$ is indexed by the marginal cell i_D , it is a function of the joint probabilities $p(i)$ in the full table.

Making the change of variables

$$(n(i), i \in \mathcal{I} \setminus \{i^*\}) \mapsto (n(i_D), D \subseteq V, i_D \in \mathcal{I}_D^*),$$

it is relatively easy to show the following expression of the multinomial distribution.

Proposition 2.1 *The NEF-representation of the saturated multinomial model in terms of the log-linear parameters $\theta(i_D)$ is given by*

$$\prod_{i \in \mathcal{I}} p(i)^{n(i)} = \exp \left\{ \sum_{D \subseteq V} \sum_{i_D \in \mathcal{I}_D^*} n(i_D) \theta(i_D) - N \log \left(1 + \sum_{D \subseteq V} \sum_{i_D \in \mathcal{I}_D^*} \exp \sum_{F \subseteq D} \theta(i_F) \right) \right\}. \quad (2.11)$$

We remark that the canonical parameter $\theta(i_D)$ in (2.10), $D \subseteq V$, is defined only for $i_D \in \mathcal{I}_D^*$, i.e. all those cell-configurations having no component equal to 0; alternatively the remaining components indexed by $i_D \in \mathcal{I} \setminus \mathcal{I}_D^*$ may be regarded as being set to zero in (2.11), and thus satisfy the usual “corner constraint” used for instance in GLIM. Furthermore, the canonical statistics $n(i_D)$ represent the marginal counts for all cells i_D , $D \subseteq V$ and $i_D \in \mathcal{I}_D^*$.

2.3 The case for G decomposable

If the multinomial model is Markov with respect to a given decomposable, non complete, graph G , it is a simple consequence of the Hammersley-Clifford theorem (see Lauritzen, 1996, p. 36 and Liu and Massam, 2007) that the model is Markov with respect to G if and only if for $i_D \in \mathcal{I}_D^*$, $D \subseteq V$

$$\theta(i_D) = 0 \text{ whenever } D \text{ is not complete in } G. \quad (2.12)$$

The model (2.11) satisfying (2.12) as the multinomial model \mathcal{M}_G Markov with respect to G . More briefly, we refer to it as the multinomial Markov model.

For any subset $A \subseteq V$ of the vertex set, define

$$\mathcal{D}^A = \{D \subseteq A \mid D \text{ is complete}\} \quad (2.13)$$

$$\mathcal{D}_0^A = \{D \subseteq_0 A \mid D \text{ is complete}\}. \quad (2.14)$$

To simplify notation, we will write \mathcal{D} for \mathcal{D}^V . We are going to present in this subsection three parametrizations for \mathcal{M}_G .

The first parametrization is in terms of the log-linear parameters defined in (2.10) with canonical parameter

$$\theta^{mod} = \theta(\mathcal{D}) = (\theta(i_D), D \in \mathcal{D}, i_D \in \mathcal{I}_D^*) \quad (2.15)$$

and corresponding canonical statistic

$$n(\mathcal{D}) = (n(i_D), D \in \mathcal{D}, i_D \in \mathcal{I}_D^*). \quad (2.16)$$

It will also be convenient to use the notation

$$k(\theta(\mathcal{D}^A)) = \log \left(1 + \sum_{D \subseteq A} \sum_{i_D \in \mathcal{I}_D^*} \exp \sum_{F \subseteq D} \theta(i_F) \right) \quad (2.17)$$

for the cumulant generating function, and the notation

$$\langle \theta(\mathcal{D}^A), n(\mathcal{D}^A) \rangle = \sum_{D \subseteq A} \sum_{i_D \in \mathcal{I}_D^*} \theta(i_D) n(i_D). \quad (2.18)$$

for the inner product. The NEF representation in terms of $\theta(\mathcal{D})$ can then be immediately derived from (2.11) as follows.

Proposition 2.2 *Let G be a decomposable graph. The NEF-representation of the multinomial Markov model in terms of the parametrization θ^{mod} is given by*

$$\exp\{\langle \theta(\mathcal{D}), n(\mathcal{D}) \rangle - N k(\theta(\mathcal{D}))\}. \quad (2.19)$$

Let us now introduce a second parametrization which is relative to the marginal distribution for C_1 and the conditional distributions for R_l given S_l . For a given perfect ordering C_1, \dots, C_k of the cliques of G , the Markov property implies (see Lauritzen, 1996, p. 90)

$$p(i) = \frac{\prod_{l=1}^k p^{C_l}(i_{C_l})}{\prod_{l=2}^k p^{S_l}(i_{S_l})}. \quad (2.20)$$

As a consequence we can write the multinomial density (2.7) as

$$\begin{aligned} \prod_{i \in \mathcal{I}} p(i)^{n(i)} &= \prod_{i \in \mathcal{I}} \left(\frac{\prod_{l=1}^k p^{C_l}(i_{C_l})}{\prod_{l=2}^k p^{S_l}(i_{S_l})} \right)^{n(i)} = \prod_{i \in \mathcal{I}} \left(p^{C_1}(i_{C_1}) \prod_{l=2}^k p^{R_l|i_{S_l}}(i_{R_l}) \right)^{n(i)} \\ &= \prod_{i_{C_1} \in \mathcal{I}_{C_1}} (p^{C_1}(i_{C_1}))^{n(i_{C_1})} \prod_{l=2}^k \prod_{i_{C_l} \in \mathcal{I}_{C_l}} (p^{R_l|i_{S_l}}(i_{R_l}))^{n(i_{C_l})} \\ &= \prod_{i_{C_1} \in \mathcal{I}_{C_1}} (p^{C_1}(i_{C_1}))^{n(i_{C_1})} \prod_{l=2}^k \prod_{i_{S_l} \in \mathcal{I}_{S_l}} \prod_{i_{R_l} \in \mathcal{I}_{R_l}} (p^{R_l|i_{S_l}}(i_{R_l}))^{n(i_{C_l})}. \end{aligned} \quad (2.21)$$

Note that (2.21) expresses the multinomial Markov model in terms of the marginal probabilities in the C_1 -table, as well as the conditional probabilities in the i_{S_l} -slice of the R_l -table, for $l = 2, \dots, k$.

Formally, for $B \subset V$ and $A \subset V$ with $A \cap B = \emptyset$, the i_B -slice of the A -table is obtained by classifying, according to the factors in A , only those units that belong

to the marginal i_B -cell (for the notion of “slice” in a contingency table see Lauritzen, 1996, p. 68).

Let us now define the log-linear parameters corresponding to the factorization (2.21).

Definition 2.3 *For each clique C_l , $l = 1, \dots, k$, we define*

$$\theta^{C_l}(i_D) = \log \prod_{F \subseteq_0 D} \left(p^{C_l}(i_F, i_{C_l \setminus F}^*) \right)^{(-1)^{|D \setminus F|}}, \quad D \subseteq C_l, \quad i_D \in \mathcal{I}_D^*. \quad (2.22)$$

Definition 2.4 *For each residual R_l , $l = 2, \dots, k$, and fixed $i_{S_l} \in \mathcal{I}_{S_l}$, we define*

$$\theta^{R_l|i_{S_l}}(i_D) = \log \prod_{F \subseteq_0 D} \left(p^{R_l|i_{S_l}}(i_F, i_{R_l \setminus F}^*) \right)^{(-1)^{|D \setminus F|}}, \quad D \subseteq R_l, \quad i_D \in \mathcal{I}_D^*. \quad (2.23)$$

Note that both $\theta^{C_l}(i_D)$ and $\theta^{R_l|i_{S_l}}(i_D)$ are “marginal” parameters, in the sense that they are functions of probabilities in the C_l -marginal table.

For any $A \subseteq V, B \subseteq V, B \cap A = \emptyset$ and any fixed $i_B \in \mathcal{I}_B$, we also introduce the notation

$$\begin{aligned} \theta(\mathcal{D}^{C_1}) &= (\theta^{C_1}(i_D), D \subseteq C_1, i_D \in \mathcal{I}_D^*), \\ n(\mathcal{D}^{C_1}) &= (n(i_D), D \subseteq C_1, i_D \in \mathcal{I}_D^*), \end{aligned} \quad (2.24)$$

representing the log-linear parameters and cell-counts for the clique- C_1 -table. Furthermore we will use

$$\begin{aligned} \theta(i_B, \mathcal{D}^A) &= (\theta^{A|i_B}(i_D), D \subseteq A, i_D \in \mathcal{I}_D^*), \\ n(i_B, \mathcal{D}^A) &= (n(i_B, i_D), D \subseteq A, i_D \in \mathcal{I}_D^*), \end{aligned} \quad (2.25)$$

to represent the log-linear parameters and the cell-counts respectively in the i_B -slice of the A -table.

We collect together the elements of (2.24) and (2.25) in a single parameter that we call θ^{cond}

$$\theta^{cond} = (\theta(\mathcal{D}^{C_1}), \theta(i_{S_l}, \mathcal{D}^{R_l}), i_{S_l} \in \mathcal{I}_{S_l}, l = 2, \dots, k). \quad (2.26)$$

Correspondingly we define the following canonical statistics

$$n^{cond} = (n(\mathcal{D}^{C_1}), \quad n(i_{S_l}, \mathcal{D}^{R_l}), \quad i_{S_l} \in \mathcal{I}_{S_l}, l = 2, \dots, k). \quad (2.27)$$

Since C_1 and $R_l, l = 2, \dots, k$ are complete, we can apply Proposition 2.1 to each of the C_1 -marginal and R_l -conditional multinomials in the i_{S_l} -slice of (2.21). We have the following lemma as an immediate consequence of Proposition 2.1.

Lemma 2.1 *The NEF-representation, in terms of the parametrization θ^{cond} ,*

- *of the marginal C_1 -model is given by*

$$\prod_{i_{C_1} \in \mathcal{I}_{C_1}} (p^{C_1}(i_{C_1}))^{n(i_{C_1})} = \exp\{\langle \theta(\mathcal{D}^{C_1}), n(\mathcal{D}^{C_1}) \rangle - N k(\theta(\mathcal{D}^{C_1}))\}. \quad (2.28)$$

- *of the conditional R_l -model in the i_{S_l} -slice is given by*

$$\prod_{i_{R_l} \in \mathcal{I}_{R_l}} \left(p^{R_l|i_{S_l}}(i_{R_l}) \right)^{n(i_{R_l})} = \exp\{\langle \theta(i_{S_l}, \mathcal{D}^{R_l}), n(i_{S_l}, \mathcal{D}^{R_l}) \rangle - n(i_{S_l}) k(\theta(i_{S_l}, \mathcal{D}^{R_l}))\} \quad (2.29)$$

Note that the number of parameters in θ^{mod} and θ^{cond} is of course the same. Indeed each element of each one of the two parametrizations is indexed by $i_D, D \in \mathcal{D}, i_D \in \mathcal{I}_D^*$ either directly as for θ^{mod} , or through the components $i_F, F \subseteq S_l, i_F \in \mathcal{I}_F^*$ and $i_D, D \subseteq R_l, i_D \in \mathcal{I}_D^*$ as for θ^{cond} .

Since the clique marginal generalized log-odds ratios are also of interest, we are now going to define a third parametrization of the multinomial model in terms of the generalized log-odds ratios in (2.22). Any marginal cell i_{S_l} can be written as

$$i_{S_l} = (i_F, i_{S_l \setminus F}^*)$$

where $F \in \mathcal{E}_{i_{S_l}}^*$. Accordingly, we define

$$\begin{aligned} \theta(\mathcal{D}_0^{S_l}, \mathcal{D}^{R_l}) &= (\theta^{C_l}(i_F, i_D), \quad D \subseteq R_l, \quad i_D \in \mathcal{I}_D^*, \quad F \subseteq_0 S_l, \quad i_F \in \mathcal{I}_F^*) \\ n(\mathcal{D}_0^{S_l}, \mathcal{D}^{R_l}) &= (n(i_F, i_D), \quad D \subseteq R_l, \quad i_D \in \mathcal{I}_D^*, \quad F \subseteq_0 S_l, \quad i_F \in \mathcal{I}_F^*) \end{aligned}$$

and

$$\theta^{cliq} = (\theta(\mathcal{D}^{C_1}), \theta(\mathcal{D}_0^{S_l}, \mathcal{D}^{R_l}), \quad l = 2, \dots, k). \quad (2.30)$$

$$n^{cliq} = (n(\mathcal{D}^{C_1}), \quad n(\mathcal{D}_0^{S_l}, \mathcal{D}^{R_l}), \quad l = 2, \dots, k). \quad (2.31)$$

We note that for $F = \emptyset$, $\theta^{C_l}(i_F, i_D) = \theta^{C_l}(i_D)$ and $n(i_F, i_D) = n(i_D)$. Clearly the number of parameters in θ^{cliq} is the same as in θ^{cond} .

The expression of the density in terms of this new parametrization will be given in the next section, after we have derived the relationship between the three parametrizations (2.15), (2.26) and (2.30).

2.4 Relationship between the various θ parametrizations

The relationship between the three θ parametrizations is given in the following proposition. To state the results succinctly, let us also define, for any $F \subseteq V$ and $i_F \in \mathcal{I}_F^*$,

$$i_{\subseteq_0 F} = \{i_G, \quad G \subseteq_0 F\}.$$

Then for given $F \subseteq V$ and $i_F \in \mathcal{I}_F^*$, and $A \subseteq V$ such that $F \cap A = \emptyset$, we also define

$$\theta(i_{\subseteq_0 F}, \mathcal{D}^A) = (\theta(i_G, j_L), \quad G \subseteq_0 F, \quad L \subseteq A, \quad j_L \in \mathcal{I}_L^*) \quad (2.32)$$

and

$$k(\theta(i_{\subseteq_0 F}, \mathcal{D}^A)) = \log \left(1 + \sum_{L \subseteq A} \exp \sum_{\substack{K \subseteq_0 F \\ H \subseteq L, \\ j_H \in \mathcal{I}_H^*}} \theta(i_K, j_H) \right). \quad (2.33)$$

We note that for any $l = 2, \dots, k$, and $F \subseteq S_l$,

$$\theta(i_{\subseteq_0 F}, \mathcal{D}^{R_l}) \subset \theta(\mathcal{D}_0^{S_l}, \mathcal{D}^{R_l}).$$

Proposition 2.3 *Let $i_D \in \mathcal{I}_D^*$ and $D \subseteq C_l, D \cap R_l \neq \emptyset$. Then*

a) the relationship between θ^{cliq} and θ^{cond} is

$$\theta^{C_l}(i_D) = \sum_{F \subseteq_0 D \cap S_l} (-1)^{|(D \cap S_l) \setminus F|} \theta^{R_l|(i_F, i_{S_l \setminus F}^*)}(i_{D \cap R_l}) \quad (2.34)$$

which, for $D \subseteq R_l$, is equivalent to

$$\theta^{R_l|(i_F, i_{S_l \setminus F}^*)}(i_D) = \sum_{G \subseteq_0 F} \theta^{C_l}(i_G, i_D) \quad (2.35)$$

b) The relationship between θ^{cliq} and θ^{mod} is as follows. Let $\{> l\}$ denote the set of $j \in \{l+1, \dots, k\}$ such that $C_l \cap C_j \neq \emptyset$.

(i) For $D \not\subseteq S_j$, for some $j \in \{> l\}$,

$$\theta(i_D) = \theta^{C_l}(i_D). \quad (2.36)$$

(ii) For $D \subseteq S_m, m \in \{> l\}$

$$\theta(i_D) = \theta^{C_l}(i_D) - \sum_{F \subseteq_0 D} (-1)^{|D \setminus F|} k(\theta(i_{\subseteq_0 F}, \mathcal{D}^{C_{>l}})) \quad (2.37)$$

where $C_{>l} = \cup_{m>l} (C_m \setminus C_l)$ and $k(\theta(i_{\subseteq_0 F}, \mathcal{D}^{C_{>l}}))$ is defined as in (2.33).

Moreover, all $\theta(i_H, j_G) \in \theta(i_{\subseteq_0 F}, \mathcal{D}^{C_{>l}})$ are such that $H \cup G \subseteq C_m$ for some $m \in \{> l\}$ and is therefore either equal to $\theta^{C_m}(i_H, j_G)$ or can be expressed in terms of $\theta^{C_m}(i_E), m \in \{> l\}, E \subseteq C_m, i_E \in \mathcal{I}_E^*$.

The proofs of (2.34) and (2.35) can easily be derived from Definitions 2.3 and 2.4. The proof of (2.37), though, is not immediate and is interesting. It is given in the appendix.

Remarks. 1. Expression (2.35) is a generalization of the relationship between conditional and marginal log-odds ratios for a three way table given in Agresti (2002, p. 322).

2. According to (2.37), $\theta(i_D)$ is a function of $\theta^{C_m}(j_H)$ such that $H \subseteq C_m$ for $m \geq l$ only. This is going to be a crucial fact when we derive the reference prior of θ^{model} from the reference prior on θ^{cond} in the next section.

Relation (2.37) is crucial for the derivation of the reference prior for θ^{cliq} in the next section and we therefore illustrate it here with an example.

Example 2.1 Consider a decomposable graphical model with the following perfect order of the cliques

$$C_1 = \{a, b, c\}, C_2 = \{b, c, d\}, C_3 = \{c, d, e\}, C_4 = \{e, f\},$$

having separators

$$S_2 = \{b, c\}, S_3 = \{c, d\}, S_4 = \{e\}.$$

To simplify matters, let us assume the data are binary. In this case we can simplify the notation since, because of the corner constraint conditions (see end of §2.2), \mathcal{I}_D^* contains only one element for each D . Thus $\theta(i_D)$ can more simply be written $\theta(D)$. Let us take $D = \{c, d\}$. We see that $D \subseteq C_2$ and $D \cap R_2 = \{d\} \neq \emptyset$. Moreover $C_{>2} = \{e, f\}$ and the set of $L \subseteq C_{>2}$ is equal to $\{e, f, ef\}$. Then according to (2.37), it follows that

$$\begin{aligned} \theta(cd) &= \theta^{C_2}(cd) - \log(1 + e^{\theta(e)+\theta(ec)+\theta(ed)+\theta(ecd)} + e^{\theta(f)} + e^{\theta(e)+\theta(ec)+\theta(ed)+\theta(ecd)+\theta(f)+\theta(ef)}) \\ &\quad + \log(1 + e^{\theta(e)+\theta(ed)} + e^{\theta(f)} + e^{\theta(e)+\theta(ed)+\theta(f)+\theta(ef)}) \\ &\quad + \log(1 + e^{\theta(e)+\theta(ec)} + e^{\theta(f)} + e^{\theta(e)+\theta(ec)+\theta(f)+\theta(ef)}) \\ &\quad - \log(1 + e^{\theta(e)} + e^{\theta(f)} + e^{\theta(e)+\theta(f)+\theta(ef)}) \end{aligned}$$

Since

$$\theta(ec) = \theta^{C_3}(ec), \theta(ed) = \theta^{C_3}(ed), \theta(ecd) = \theta^{C_3}(ecd), \theta(ef) = \theta^{C_4}(ef), \theta(f) = \theta^{C_4}(f),$$

and according to (2.37) again,

$$\theta(e) = \theta^{C_3}(e) + \log(1 + e^{\theta(f)}) - \log(1 + e^{\theta(f)+\theta(ef)}) = \theta^{C_3}(e) + \log(1 + e^{\theta^{C_4}(f)}) - \log(1 + e^{\theta^{C_4}(f)+\theta^{C_4}(ef)}),$$

we see that $\theta(cd)$ can be expressed in terms of $\theta^{C_m}(E)$, $m \geq 2$, $E \subseteq C_m$.

We will now give the expression of the multinomial Markov model with respect to θ^{cliq} , using relation (2.35).

Lemma 2.2 *Let G be a decomposable graph with its cliques C_1, \dots, C_k arranged in a perfect order. The NEF-representation of the multinomial Markov model in terms of the θ^{cliq} parametrization is given by*

$$\begin{aligned} \prod_{i \in \mathcal{I}} p(i)^{n(i)} &= \exp\{\langle \theta(\mathcal{D}^{C_1}), n(\mathcal{D}^{C_1}) \rangle - N k(\theta(\mathcal{D}^{C_1}))\} \\ &\quad \times \prod_{l=2}^k \exp\{\langle \theta(\mathcal{D}_0^{S_l}, \mathcal{D}^{R_l}), n(\mathcal{D}_0^{S_l}, \mathcal{D}^{R_l}) \rangle \\ &\quad - \sum_{F \subseteq_0 S_l} \sum_{j_F \in \mathcal{I}_F^*} n(j_F) \sum_{H \subseteq_0 F} (-1)^{|F \setminus H|} k(\theta(j_{\subseteq_0 H}, \mathcal{D}^{R_l}))\} \end{aligned} \quad (2.38)$$

From (2.38), it appears that under the multinomial Markov model, the joint distribution of n^{cliq} admits a conditional reducibility structure, see Consonni and Veronese (2001); specifically, it factorizes into the product of k conditional exponential families (save for the first term which is a marginal distribution), in a recursive fashion according to the clique ordering.

3 Reference priors

In this section we shall derive reference priors for the various parametrizations introduced in section 2. We shall only provide an outline of the proofs of the derivation of our reference priors since they follow the steps described in §2 and §4.2.1 of Consonni et al. (2004). An important point to keep in mind is that a reference prior for a multidimensional parameter depends on the grouping of its components, as well as the ordering of its groups: specifically we order groups according to inferential importance, while parameter-components that belong to the same group are treated in a symmetric fashion. For the parametrizations considered in this paper, order will not matter, and thus the reference prior will only depend on the grouping-structure.

For a given graph G , let C_1, \dots, C_k represent a perfect ordering of the cliques. We will first consider the reference prior for the collection of conditional probabilities (including the marginal probabilities for clique C_1), p^{cond} as in (2.21)

$$p^{cond} = (p^{C_1}, p^{R_l|i_{S_l}}, i_{S_l} \in \mathcal{I}_{S_l}, l = 2, \dots, k), \quad (3.1)$$

where

$$p^{C_1} = (p^{C_1}(i_{C_1}), i_{C_1} \in \mathcal{I}_{C_1}) \quad (3.2)$$

$$p^{R_l|i_{S_l}} = (p^{R_l|i_{S_l}}(i_{R_l}), i_{R_l} \in \mathcal{I}_{R_l}), \quad (3.3)$$

represent the collection of groups. Note that there are $1 + \sum_{l=2}^k |\mathcal{I}_{S_l}|$ groups. We remark that the nature of the parametrization p^{cond} depends on the specific choice of the perfect numbering of the cliques C_1, \dots, C_k .

Next we will consider the reference priors for $\theta^{cond}, \theta^{cliq}, \theta^{mod}$ following a parallel grouping-structure. We shall see that all these reference priors are strictly related, so that a unified expression for all of them is possible.

Proposition 3.1 *The reference prior for p^{cond} relative to the grouping defined in (3.1) is*

$$\pi_{p^{cond}}^R(p^{cond}) \propto \left(\prod_{i_{C_1} \in \mathcal{I}_{C_1}} p(i_{C_1}) \right)^{-\frac{1}{2}} \prod_{l=2}^k \prod_{i_{S_l} \in \mathcal{I}_{S_l}} \left(\prod_{i_{R_l} \in \mathcal{I}_{R_l}} p^{R_l|i_{S_l}}(i_{R_l}) \right)^{-\frac{1}{2}}. \quad (3.4)$$

We note that the reference prior is a product of Jeffreys' priors, one for each of the groups of p^{cond} .

Proof: In our setting, we simply need to derive the (Fisher) information matrix. From (2.21) it appears that the likelihood function factorizes into the product of terms, each involving exactly one group of p^{cond} ; furthermore each term is a saturated multinomial. Accordingly the information matrix is block-diagonal, and the determinant of each block, using classic results, is easily available. Specifically the first one, corresponding to clique C_1 , is given by

$$N\left(\prod_{i_{C_1} \in \mathcal{I}_{C_1}} p(i_{C_1}) \right)^{-1}, \quad (3.5)$$

while for the remaining blocks the determinant is

$$E(n(i_{S_l})|p) \left(\prod_{i_{R_l} \in \mathcal{I}_{R_l}} p^{R_l|i_{S_l}}(i_{R_l}) \right)^{-1}, \quad i_{S_l} \in \mathcal{I}_{S_l}, l = 2, \dots, k. \quad (3.6)$$

Because of the perfect ordering the cliques, $S_l \subseteq C_j$ for some $j < l$, so that the expected value $E(n(i_{S_l})|p)$ is a function of parameters only belonging to groups preceding the l -th one.

Following the theory summarized in Consonni et al. (2004, sect. 2), the reference prior is given by the square root of the product of the block-determinants, excluding the terms $E(n(i_{S_l})|p)$, and the result is established. \square

We now emphasize three properties of the reference prior for p^{cond} . First of all, since the information matrix is block-diagonal, the reference prior is order-invariant,

i.e. it does not depend on the order of the groups. Secondly, we remark that there exists also some degree of invariance with respect to grouping. Specifically, if we lumped together in one single block all the i_{S_l} terms $p^{R_l|i_{S_l}}$, $i_{S_l} \in \mathcal{I}_{S_l}$, the reference prior would not change. This feature will turn out to be useful later on when deriving reference priors for alternative parametrizations.

Third, we remark that the distribution $\pi_{p^{cond}}^R$ belongs to a family conjugate to the likelihood for p^{cond} , see (2.21). Accordingly its hyper-parameters can be interpreted in terms of “prior counts”; the latter however cannot be recovered as the margins of an fictitious overall table. Indeed, each cell in the C_1 -table, as well as in the i_{S_l} slice of the R_l -table, has a prior count equal to $1/2$, irrespective of the dimension of the subtables and of the overall table. Finally, the prior is proper, since it is a product of Dirichlet priors, one for each block, each Dirichlet being indexed by a vector of hyper-parameters with entries all equal to $1/2$.

We now turn to the derivation of the reference priors for the three θ parametrizations described in §2. Central to our arguments below is the following basic fact about reference priors that we shall use repeatedly. Let λ be a parameter grouped into components $\lambda = (\lambda_1, \dots, \lambda_k)$, where λ_i is typically a vector. We assume that the above groups are arranged in increasing order of inferential importance. Let $\phi = (\phi_1, \dots, \phi_k)$ be a reparametrization, i.e. a one-to-one function of λ with ϕ_i having the same dimension as λ_i . Suppose that, for each $l = 1, \dots, k$, $\phi_l = h_l(\lambda_1, \dots, \lambda_l)$, for some function h_l : we say in this case that the map $\lambda \mapsto \phi$ is block-lower triangular. Then the reference prior for ϕ can be obtained from that of λ simply by a change of variable. For details and references see again Consonni et al. (2004, §4.2.1) An important special case occurs when $\phi_l = h_l(\lambda_l)$: in this case we say that the map is block-wise one-to-one.

We start by expressing p^{cond} in terms of the θ^{cond} . In order to achieve this goal, it is convenient to define the parameters

$$\xi^{C_1}(i_D) = \sum_{F \subseteq D} \theta^{C_1}(i_F), \quad i_D \in \mathcal{I}_D^* \quad (3.7)$$

$$\xi^{R_l|i_F}(i_F, i_D) = \sum_{L \subseteq D} \theta^{R_l|(i_F, i_{S_l \setminus F}^*)}(i_L) \quad (3.8)$$

$$= \sum_{L \subseteq D} \sum_{H \subseteq_0 F} \theta^{C_l}(i_H, i_L), \quad i_F \in \mathcal{I}_F^*, i_L \in \mathcal{I}_L^*, D \subseteq R_l \quad (3.9)$$

We let

$$\xi = (\xi^{C_1}, \xi^{R_l|(i_F, i_{S_l \setminus F}^*)}, F \subseteq S_l, i_F \in \mathcal{I}_F^*, l = 2, \dots, k) \quad (3.10)$$

where

$$\begin{aligned} \xi^{C_1} &= (\xi^{C_1}(i_D), D \subseteq C_1) \\ \xi^{R_l|(i_F, i_{S_l \setminus F}^*)} &= (\xi^{R_l|(i_F, i_{S_l \setminus F}^*)}(i_F, i_D), D \subseteq R_l, i_D \in \mathcal{I}_D^*), \end{aligned}$$

The mapping between p^{cond} and ξ is block-wise one-to-one. As a consequence the reference prior on ξ can be deduced from that of p^{cond} as

$$\pi_\xi^R(\xi) = \pi_{p^{cond}}^R(p^{cond}(\xi)) |J_{p^{cond}}(\xi)| \quad (3.11)$$

where $J_{p^{cond}}(\xi)$ is the Jacobian of the transformation $p^{cond} \mapsto \xi$. It can be verified that

$$\det \left(\frac{dp^{cond}}{d\xi} \right) = \left(\prod_{i_{C_1} \in \mathcal{I}_{C_1}} p(i_{C_1})(\xi^{C_1}) \right) \prod_{l=2}^k \prod_{i_{S_l} \in \mathcal{I}_{S_l}} \left(\prod_{i_{R_l} \in \mathcal{I}_{R_l}} p^{R_l|i_{S_l}}(i_{R_l})(\xi^{R_l|i_{S_l}}) \right) \quad (3.12)$$

so that the induced reference prior for ξ is

$$\pi_\xi^R(\xi) \propto \left(\prod_{i_{C_1} \in \mathcal{I}_{C_1}} p(i_{C_1})(\xi^{C_1}) \right)^{-\frac{1}{2}+1} \prod_{l=2}^k \prod_{i_{S_l} \in \mathcal{I}_{S_l}} \left(\prod_{i_{R_l} \in \mathcal{I}_{R_l}} p^{R_l|i_{S_l}}(i_{R_l})(\xi^{R_l|i_{S_l}}) \right)^{-\frac{1}{2}+1} \quad (3.13)$$

We shall also need the following result which can be easily derived from Definitions (2.3) and (2.4) and Moebius inversion formula.

Lemma 3.1 For $i_{C_1} = (i_F, i_{C_1 \setminus F}^*)$,

$$p^{C_l}(i_{C_l}) = \frac{\exp \xi^{C_l}(i_F)}{1 + \sum_{H \subseteq C_1} \sum_{j_H \in \mathcal{I}_H^*} \exp \xi_D^{C_l}(j_H)}, \quad (3.14)$$

For i_{S_l} and $i_{R_l} = (i_G, i_{R_l \setminus G}^*)$ given,

$$p^{R_l|i_{S_l}}(i_{R_l}) = \frac{\exp \xi^{R_l|i_{S_l}}(i_G)}{1 + \sum_{H \subseteq R_l} \sum_{j_H \in \mathcal{I}_H^*} \exp \xi_D^{R_l|i_{S_l}}(j_H)}, \quad (3.15)$$

As particular cases, we have

$$p^{C_l}(i_{C_l}^*) = \frac{1}{1 + \sum_{H \subseteq C_l} \sum_{j_H \in \mathcal{I}_H^*} \exp \xi_D^{C_l}(j_H)} , \quad (3.16)$$

and

$$p^{R_l|i_{S_l}}(i_{R_l}^*) = \frac{1}{1 + \sum_{H \subseteq R_l} \sum_{j_H \in \mathcal{I}_H^*} \exp \xi_D^{R_l|i_{S_l}}(j_H)} \quad (3.17)$$

Since the reference priors of the three θ -parametrizations are structurally equivalent we shall provide the result in a unified statement.

Theorem 3.1 *The reference prior for*

- a) θ^{cond} , relative to the grouping defined in (2.26)
- b) θ^{cliq} , relative to the grouping defined in (2.30)
- c) θ^{mod} , relative to the following grouping

$$\tilde{\theta}^{C_1} = (\theta(i_D), D \subseteq C_1, i_D \in \mathcal{I}_D^*), \tilde{\theta}^{C_l} = (\theta(i_D), D \subseteq C_l, D \cap R_l \neq \emptyset), l = 2, \dots, k. \quad (3.18)$$

is proportional to

$$\left(\prod_{i_{C_1} \in \mathcal{I}_{C_1}} p(i_{C_1})(\cdot) \right)^{\frac{1}{2}} \prod_{l=2}^k \prod_{i_{S_l} \in \mathcal{I}_{S_l}} \left(\prod_{i_{R_l} \in \mathcal{I}_{R_l}} p^{R_l|i_{S_l}}(i_{R_l})(\cdot) \right)^{\frac{1}{2}}, \quad (3.19)$$

where the probabilities $p(i_{C_1})(\cdot)$ and $p^{R_l|i_{S_l}}(i_{R_l})(\cdot)$ are understood to be expressed in terms of the relevant θ -parametrization, using (3.14)-(3.17) together with i) (3.7)-(3.8) for θ^{cond} ; ii) (3.9) for θ^{cliq} . iii) (2.35), (2.36) and (2.37) for θ^{mod} .

More explicitly, the reference prior

- for θ^{cond} is given by the product of (2.28) and (2.29), with the understanding that the counts in these formulas are replaced by fictitious prior counts which we write as $\tilde{n}(i_D)$, \tilde{N} and so on. More precisely, we have

$$\tilde{n}(i_D) = \frac{|\mathcal{I}_{C_1 \setminus D}|}{2}, \quad \tilde{N} = \frac{|\mathcal{I}_{C_1}|}{2},$$

and

$$\tilde{n}(i_{S_l}, i_D) = \frac{|\mathcal{I}_{R_l \setminus D}|}{2}, \quad \tilde{n}(i_{S_l}) = \frac{|\mathcal{I}_{R_l}|}{2}.$$

- for θ^{cliq} is given by (2.38) where for $l = 1$

$$\tilde{n}(i_D) = \frac{|\mathcal{I}_{C_1 \setminus D}|}{2} \quad \text{and} \quad \tilde{N} = \frac{|\mathcal{I}_{C_1}|}{2},$$

and for $l = 2, \dots, k$

$$\tilde{n}(j_F, i_D) = \frac{|\mathcal{I}_{S_l \setminus F}| |\mathcal{I}_{R_l \setminus D}|}{2} \quad \text{and} \quad \tilde{n}(j_F) = \frac{|\mathcal{I}_{S_l \setminus F}| |\mathcal{I}_{R_l}|}{2}$$

- for θ^{mod} can be obtained from that of θ^{cliq} above by expressing it in terms of $\theta(\mathcal{D})$ using (2.36) and (2.37).

Proof: a) Because of (3.7) and (3.8) it is immediate to verify that the map $\xi \mapsto \theta^{cond}$ is block-wise one-to-one; moreover the Jacobian is equal to one. Accordingly the reference prior for θ^{cond} will be exactly as that for ξ , with the only difference that the probabilities involved will be expressed as functions of θ^{cond} .

b) Similarly to what happened for the reference prior for p^{cond} , the reference prior for θ^{cond} is unchanged if, for each $l = 2, \dots, k$, we lump together the groups labeled by $i_{S_l} \in \mathcal{I}_{S_l}$, and thus only regard θ^{cond} as made up of k groups. In this way the transformation from θ^{cond} to θ^{cliq} is block-wise one-to-one, and thus the reference prior for θ^{cliq} is equal to that induced from the reference prior θ^{cond} . Moreover, the transformation is linear so that the Jacobian is constant, and thus the result follows.

c) We see that the groupings in (3.18) are exactly parallel to those in θ^{cliq} . From (2.37) we also see that the l -th group in θ^{mod} is a function of the subsequent $l, l+1, \dots, k$ groups in θ^{cliq} . This defines a block-upper triangular transformation, which can be turned into a block-lower triangular one by reversing the order of the groups in θ^{cliq} . Since the reference prior on θ^{cliq} is invariant to group-ordering, we conclude that the reference prior on θ^{mod} can be obtained from that of θ^{cliq} by a change-of-variable. Now notice that the Jacobian is 1, again using (2.37), so that the result is proved. Finally, the expressions of the fictitious counts are derived by inspection.

□

We remark that, similarly to what happened for p^{cond} , the reference prior for each of the three θ -parametrizations is also a conjugate prior and is proper, being the transformation of a proper prior on p^{cond} .

4 Parametrizations and reference priors associated to a cut

The reference priors obtained in the previous section were based on a grouping of the parameters defined by the structure of the graph, essentially through a perfect order of the cliques (and consequently of residuals and separators).

Now suppose we are interested in a particular subset $A \subseteq V$ of the variables, and that we would like to consider a reference prior which groups together precisely the parameters of the marginal distribution referring to A . We show in this section how this can be done if the Markov model \mathcal{M}_G is *collapsible* onto A , equivalently if A represents a *cut* for the joint distribution.

Asmussen and Edwards (1983) consider the concept of collapsibility for contingency tables. If the set of factors for the table are indexed by $\gamma \in V$ and if $A \subseteq V$, we say that G is collapsible onto A if the multinomial model \mathcal{M}_{G_A} , Markov with respect to the induced subgraph G_A , is the same as the model obtained by marginalising the given model \mathcal{M}_G , Markov with respect to G , over the A -table.

Frydenberg (1990, Theorem 5.4) has shown that the model for the random vector Y , Markov with respect to G , is collapsible onto A if and only if the sub-vector Y_A is a cut (for simplicity we shall also say that A is a cut). Cuts in exponential families have been introduced in Barndorff-Nielsen (1978) and studied in several further articles such as Barndorff-Nielsen and Koudou (1995).

A very useful result, due to Asmussen and Edwards (1983), is that A will induce a cut if and only if the boundary of every connected component of $V \setminus A$ has a complete boundary in G .

The following lemma gives us the factorization of \mathcal{M}_G with respect to the cut A and the connected components of $G_{V \setminus A}$.

Lemma 4.1 *Let A be a cut. Let C'_1, \dots, C'_q be a perfect ordering of the cliques of G_A , the graph induced by A . Let $B_l, l = 1, \dots, p$ be the connected components of $G_{V \setminus A}$. Let $C_j^l, j = 1, \dots, m_l$ be the cliques of the induced graph $G_{B_l \cup \partial B_l}, l = 1, \dots, p$. The*

multinomial model \mathcal{M}_G , Markov with respect to G , can be factorized as follows

$$\begin{aligned} \prod_{i \in \mathcal{I}} p(i)^{n(i)} &= \prod_{i_{C'_1} \in \mathcal{I}_{C'_1}} (p^{C'_1}(i_{C'_1}))^{n(i_{C'_1})} \prod_{l=1}^q \prod_{i_{S'_l} \in \mathcal{I}_{S'_l}} \prod_{i_{R'_l} \in \mathcal{I}_{R'_l}} (p^{R'_l | i_{S'_l}}(i_{R'_l}))^{n(i_{C'_l})} \\ &\times \prod_{l=1}^p \left(\prod_{i_{\partial B_l}} \prod_{i_{C_1^{(l)} \setminus \partial B_l}} (p^{C_1^{(l)} \setminus \partial B_l | i_{\partial B_l}}(i_{C_1^{(l)} \setminus \partial B_l}))^{n(i_{C_1^{(l)}})} \prod_{j=2}^{m_l} \prod_{i_{S_j^{(l)}}} \prod_{i_{R_j^{(l)}}} (p^{R_j^{(l)} | i_{S_j^{(l)}}}(i_{R_j^{(l)}}))^{n(i_{C_j^{(l)}})} \right) \end{aligned} \quad (4.1)$$

Proof: For simplicity of exposition, some statements concerning the random variables associated to a set, will be simply stated in terms of the set itself. If A is a cut, A separates the connected components of $V \setminus A$; by Theorem 2.8 of Dawid and Lauritzen (1993), this implies that the B_l 's are mutually conditionally independent given A . Moreover since A is a cut, the boundary of B_l is a complete subset of A and, of course, it separates B_l from $V \setminus (B_l \cup \partial B_l)$. Therefore the overall multinomial Markov model factorizes as the product of the A -marginal multinomial model, Markov with respect to \mathcal{M}_{G_A} , and the product of the conditional multinomial distributions of the B_l s given $i_{\partial B_l}$, $l = 1, \dots, p$.

Since the the marginal model for A is Markov with respect to the graph G_A , it factorizes according to a perfect order of the cliques of G_A , in parallel to what was done in §3: this proves the first line of (4.1).

Let us now consider the expression for the second line of (4.1). As recalled above, this is given by the product of the conditional multinomial models for B_l , $l = 1, \dots, p$ given $i_{\partial B_l}$. For any $l \in \{1, \dots, p\}$, as a subgraph of G , the induced graph $G_{B_l \cup \partial B_l}$ is decomposable. Moreover the marginal model for $B_l \cup \partial B_l$ is Markov w.r.t. $G_{B_l \cup \partial B_l}$. This happens because $B_l \cup \partial B_l$ is itself a cut, since the boundary of each connected component of $G_{V \setminus (B_l \cup \partial B_l)}$ clearly belongs to ∂B_l which is complete. Therefore the marginal distribution $\mathcal{M}_{G_{B_l \cup \partial B_l}}$ factorizes according to a perfect order of the cliques of $G_{B_l \cup \partial B_l}$. Since ∂B_l is complete, it must belong to a clique $C_1^{(l)}$ of $G_{B_l \cup \partial B_l}$ and by Proposition 2.29 of Lauritzen (1996), we know that we can take this clique as the first in a perfect order $C_i^{(l)}$, $i = 1, \dots, m_l$ of the cliques of $G_{B_l \cup \partial B_l}$.

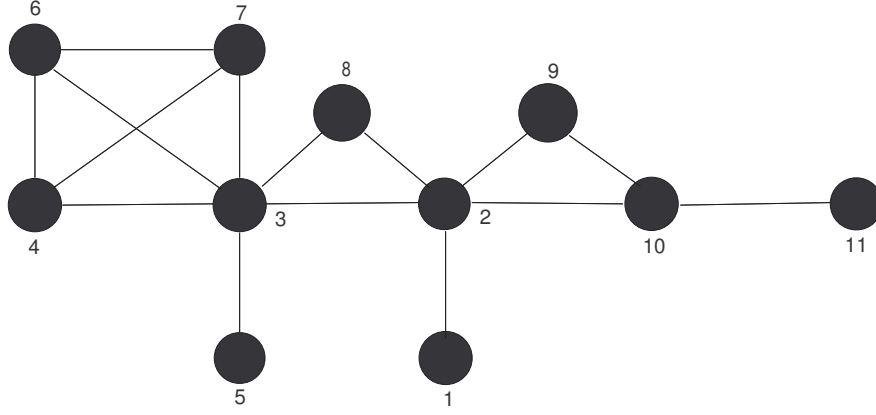


Figure 1: The decomposable graph for Example 4.1

The marginal multinomial distribution $\mathcal{M}_{G_{B_l \cup \partial B_l}}$ can therefore be written as

$$\begin{aligned}
\prod_{j=1}^{m_l} \prod_{i_{C_j^{(l)}}} (p_{C_j^{(l)}}^{C_j^{(l)}}(i_{C_j^{(l)}}))^{n(i_{C_j^{(l)}})} &= \prod_{i_{C_1^{(l)}}} p_{C_1^{(l)}}^{C_1^{(l)}}(i_{C_1^{(l)}})^{n(i_{C_1^{(l)}})} \prod_{j=2}^{m_l} \prod_{i_{S_j^{(l)}}} \prod_{i_{R_j^{(l)}}} (p_{S_j^{(l)}}^{R_j^{(l)}|i_{S_j^{(l)}}}(i_{R_j^{(l)}}))^{n(i_{C_j^{(l)}})} \\
&= \prod_{i_{\partial B_l}} \left((p^{\partial B_l}(i_{\partial B_l}))^{n(i_{\partial B_l})} \prod_{i_{C_1^{(l)} \setminus \partial B_l}} (p_{C_1^{(l)} \setminus \partial B_l}^{C_1^{(l)}|i_{\partial B_l}}(i_{C_1^{(l)} \setminus \partial B_l}))^{n(i_{C_j^{(l)}})} \right) \\
&\quad \times \prod_{j=2}^{m_l} \prod_{i_{R_j^{(l)}}} \prod_{i_{S_j^{(l)}}} p_{S_j^{(l)}}^{R_j^{(l)}|i_{S_j^{(l)}}}(i_{R_j^{(l)}})^{n(i_{C_j^{(l)}})}
\end{aligned}$$

and therefore the model for B_l conditional on $i_{\partial B_l}$ is equal to

$$\prod_{i_{C_1^{(l)} \setminus \partial B_l}} (p_{C_1^{(l)} \setminus \partial B_l}^{C_1^{(l)}|i_{\partial B_l}}(i_{C_1^{(l)} \setminus \partial B_l}))^{n(i_{C_j^{(l)}})} \prod_{j=2}^{m_l} \prod_{i_{R_j^{(l)}}} \prod_{i_{S_j^{(l)}}} p_{S_j^{(l)}}^{R_j^{(l)}|i_{S_j^{(l)}}}(i_{R_j^{(l)}})^{n(i_{C_j^{(l)}})}. \quad (4.2)$$

Since this is true for all B_l , the result is established. \square

Example 4.1 Suppose that the joint distribution of the 11 variables numbered consecutively from 1 to 11 is Markov with respect to the decomposable graph G as given in Fig. 1.

Consider the subset of variables given by $A = \{1, 2, 3, 4\}$. A perfect ordering of the cliques of the induced sub-graph G_A is

$$C'_1 = \{1, 2\}, C'_2 = \{2, 3\}, C'_3 = \{3, 4\}, \quad (4.3)$$

so that $S'_2 = \{2\}$, $S'_3 = \{3\}$, $R'_2 = \{3\}$, $R'_3 = \{4\}$. The connected components B_l of $G_{V \setminus A}$, their boundary ∂B_l together with the cliques C_j^l of $G_{B_l \cup \partial B_l}$ are

| l | B_l | ∂B_l | $B_l \cup \partial B_l$ | $C_j^{(l)}$ |
|-----|-----------------|----------------|-------------------------|---|
| 1 | $\{9, 10, 11\}$ | $\{2\}$ | $\{2, 9, 10, 11\}$ | $C_1^{(1)} = \{2, 9, 10\}$, $C_2^{(1)} = \{10, 11\}$ |
| 2 | $\{8\}$ | $\{2, 3\}$ | $\{2, 3, 8\}$ | $C_1^{(2)} = \{2, 3, 8\}$ |
| 3 | $\{5\}$ | $\{3\}$ | $\{3, 5\}$ | $C_1^{(3)} = \{3, 5\}$ |
| 4 | $\{6, 7\}$ | $\{3, 4\}$ | $\{3, 4, 6, 7\}$ | $C_1^{(4)} = \{3, 4, 6, 7\}$ |

A graphical display of G_A and its connected components is given in Fig. 2.

Accordingly, the multinomial model, Markov with respect to G , can be factorized using Lemma 4.1 as

$$\begin{aligned}
\prod_{i \in \mathcal{I}} p(i)^{n(i)} &= \\
&\prod_{i_{C'_1} \in \mathcal{I}_{C'_1}} (p^{C'_1}(i_{C'_1}))^{n(i_{C'_1})} \prod_{i_{S'_2} \in \mathcal{I}_{S'_2}} \prod_{i_{R'_2} \in \mathcal{I}_{R'_2}} (p^{R'_2|i_{S'_2}}(i_{R'_2}))^{n(i_{C'_2})} \prod_{i_{S'_3} \in \mathcal{I}_{S'_3}} \prod_{i_{R'_3} \in \mathcal{I}_{R'_3}} (p^{R'_3|i_{S'_3}}(i_{R'_3}))^{n(i_{C'_3})} \\
&\times \prod_{i_{\partial B_1}} \prod_{i_{C_1^{(1)} \setminus \partial B_1}} (p^{C_1^{(1)} \setminus \partial B_1 | i_{\partial B_1}}(i_{C_1^{(1)} \setminus \partial B_1}))^{n(i_{C_1^{(1)}})} \prod_{i_{S_2^{(1)}}} \prod_{i_{R_2^{(1)}}} (p^{R_2^{(1)} | i_{S_2^{(1)}}}(i_{R_2^{(1)}}))^{n(i_{C_2^{(1)}})} \\
&\times \prod_{i_{\partial B_2}} \prod_{i_{C_1^{(2)} \setminus \partial B_2}} (p^{C_1^{(2)} \setminus \partial B_2 | i_{\partial B_2}}(i_{C_1^{(2)} \setminus \partial B_2}))^{n(i_{C_1^{(2)}})} \\
&\times \prod_{i_{\partial B_3}} \prod_{i_{C_1^{(3)} \setminus \partial B_3}} (p^{C_1^{(3)} \setminus \partial B_3 | i_{\partial B_3}}(i_{C_1^{(3)} \setminus \partial B_3}))^{n(i_{C_1^{(3)}})} \\
&\times \prod_{i_{\partial B_4}} \prod_{i_{C_1^{(4)} \setminus \partial B_4}} (p^{C_1^{(4)} \setminus \partial B_4 | i_{\partial B_4}}(i_{C_1^{(4)} \setminus \partial B_4}))^{n(i_{C_1^{(4)}})}
\end{aligned}$$

We now provide the expression for the reference prior associated to a cut.

Theorem 4.1 *Let A be a cut and consider the parametrization associated to A*

$$p_A^{cut} = (p_A^{t, cond}, p_{V \setminus A | A}^{t, cond}),$$

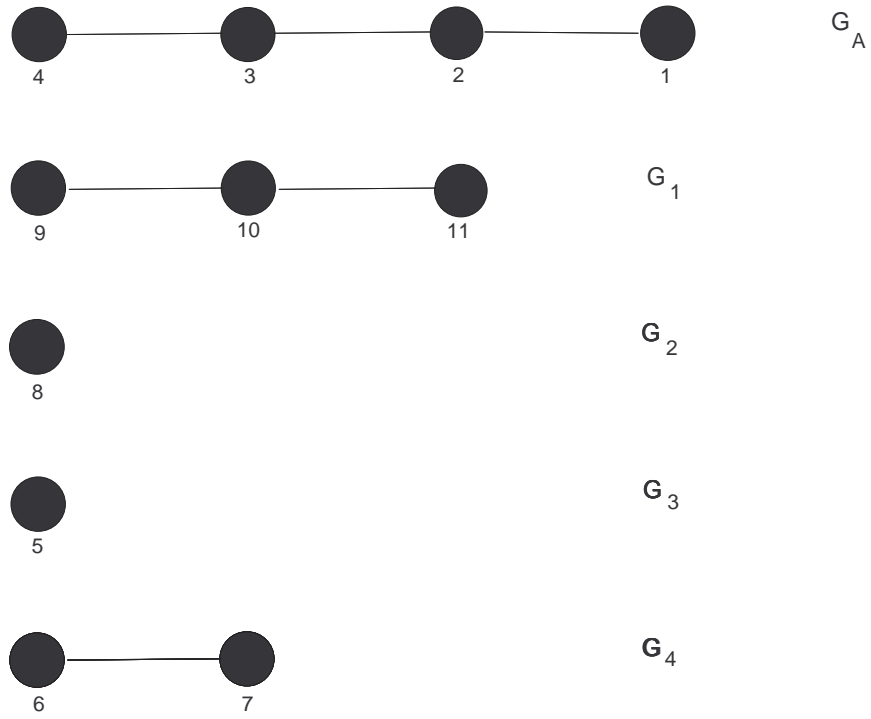


Figure 2: The decomposable graph G_A associated to a cut A and the connected components of $G_{V \setminus A}$ for Example 4.1

where

$$p_A'^{cond} = (p^{C'_1}, p^{R'_l|i_{S'_l}}, l = 1, \dots, q, i_{S'_l} \in \mathcal{I}_{S'_l}) \quad (4.4)$$

$$p_{V \setminus A|A}'^{cond} = (p^{C_1^{(l)} \setminus \partial B_l | i_{\partial B_l}}, i_{\partial B_l} \in \mathcal{I}_{\partial B_l}; p^{R_j^{(l)} | i_{S_j^{(l)}}}, l = 1, \dots, p, \\ j = 2, \dots, m_l, i_{S_j^{(l)}} \in \mathcal{I}_{S_j^{(l)}}), \quad (4.5)$$

using the notation presented in Lemma 4.1. The reference prior for p_A^{cut} , relative to the grouping (4.4) and (4.5), is

$$\pi_{p_A^{cut}}^R(p_A^{cut}) \propto \prod_{i_{C'_1} \in \mathcal{I}_{C'_1}} p^{C'_1}(i_{C'_1})^{-\frac{1}{2}} \prod_{l=1}^q \prod_{i_{S'_l} \in \mathcal{I}_{S'_l}} \prod_{i_{R'_l} \in \mathcal{I}_{R'_l}} (p^{R'_l|i_{S'_l}}(i_{R'_l}))^{-\frac{1}{2}} \\ \times \prod_{l=1}^p \left(\prod_{i_{\partial B_l}} \prod_{i_{C_1^{(l)} \setminus \partial B_l}} (p^{C_1^{(l)} \setminus \partial B_l | i_{\partial B_l}}(i_{C_1^{(l)} \setminus \partial B_l}))^{-\frac{1}{2}} \prod_{j=2}^{m_l} \prod_{i_{S_j^{(l)}}} \prod_{i_{R_j^{(l)}}} (p^{R_j^{(l)} | i_{S_j^{(l)}}}(i_{R_j^{(l)}}))^{-\frac{1}{2}} \right)$$

We emphasize that, also for this case, the prior admits a conjugate structure and is proper, being a product of Jeffreys' priors.

Proof: Using Lemma 4.1 the likelihood factorizes into a product of two general terms, one related to the marginal distribution of A indexed by $p_A'^{cond}$, the other related to the conditional distribution of $V \setminus A$ given A indexed by $p_{V \setminus A|A}'^{cond}$. The two groups of parameters are variation and likelihood independent, so that the information matrix is two-block-diagonal. The marginal distribution related to A is a G_A -Markov model, with G_A decomposable, and therefore the corresponding reference prior is exactly as in the general decomposable case of Proposition 3.1. This yields the first line of the kernel of the reference prior

To prove the second line, we have to consider the second block of the information matrix. This actually further decomposes into p diagonal blocks, one for each connected component B_l . Consider the block corresponding to the model for B_l conditional on $\partial B_l, l = 1, \dots, p$ (see (4.2)). Each block represents the information of a saturated conditional multinomial. In particular the first term has cell-probabilities $p^{C_1^{(l)} \setminus \partial B_l | i_{\partial B_l}}$ and $n(i_{\partial B_l})$ trials, while the remaining terms have cell-probabilities $p^{R_j^{(l)} | i_{S_j^{(l)}}}$ and $n(i_{S_j^{(l)}})$ trials. The expression of the corresponding term in

the information matrix will therefore be as in the general conditional saturated multinomial, see (3.6). Finally, the expectation of $n(i_{\partial B_l})$ depends only on the parameter $p_A^{',cond}$ since $\partial B_l \in A$, and similarly the expectation of $n(i_{S_j^{(l)}})$ does not depend on the parameter $p^{R_j^{(l)}|i_{S_j^{(l)}}}$ specific to the component because of the perfect ordering of the cliques. Therefore, in both cases the term corresponding to the expectation factors out of the determinant and the proof is complete. \square

5 Discussion

In this paper we have considered several alternative parametrizations for discrete decomposable graphical models. First of all we have described a parametrization in terms of conditional cell-probabilities. Next we have derived three alternative representations in terms of natural exponential families, whose canonical parameters represent generalized log-odds ratios relative to suitable cell-probabilities. Specifically, θ^{mod} refers to the joint probabilities of the full table and has been previously used, see e.g. Dellaportas and Forster (1999), Dellaportas and Tarantola (2005) and Liu and Massam (2007) but we think that our derivation and interpretation makes its interpretation clearer. The parametrizations θ^{cond} and θ^{clique} , on the other hand, refer to marginal sub-tables and are quite distinct from those traditionally employed in graphical log-linear modelling. Indeed they are rather related to the concept of marginal models, see e.g. Bergsma and Rudas (2002), Lang and Agresti (1994) and Glonek and McCullagh (1995).

A reference prior for each of the above parametrizations was constructed. In particular the prior for the conditional cell-probabilities of the residuals given the separators is a product, of Jeffreys' priors. We showed that all reference priors are coherent, i.e. each is equivalent to any other one. This happens because the grouping structure is such that the transformation between any two parametrizations is either block-diagonal or block-lower triangular. A notable feature is that all reference priors are proper. Another property is that they belong to a conjugate family, which facili-

tates prior-to-posterior updating. The conjugacy feature is consistent with previous results, see Consonni et al. (2004), wherein reference priors for suitable parametrization of NEFs having a simple quadratic variance (such as the saturated multinomial) were derived and shown to belong to (enriched) conjugate families. Our paper shows that this result continues to hold also for multinomial decomposable models, whose variance function is not quadratic. With hindsight, this is not surprising, because of the recursive factorization into products of conditional saturated multinomial models that holds when G is decomposable. We have also considered a parametrization, and the corresponding reference prior, associated to a cut. This can be especially useful whenever interest focuses on the parameters of a marginal table, e.g. because of their inferential interest.

Throughout the paper we assumed a given graphical model, and constructed reference priors essentially in view of estimation purposes. While we are aware that estimation-based priors should not be routinely used in model determination, we remark that our reference priors are conjugate (and thus decompose into local blocks precisely like the likelihood) and that they are proper. These attractive features make them natural candidates also for model comparison, e.g. *via* Bayes factors, at least for a preliminary and informal evaluation.

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6 References

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A Appendix

Proof of (2.37). Consider $D \subseteq C_l, D \cap R_l \neq \emptyset$ for some $l \in \{1, \dots, k-1\}$ such that also $D \subseteq S_j$ for some $j \in \{> l\}$, then

$$\begin{aligned}
 p^{C_l}(i_D) &= \sum_{L \subseteq_0 C_l^c} p(i_D, j_L) = \sum_{L \subseteq_0 C_l^c} \exp \left(\sum_{E \subseteq_0 D} \theta(i_E) + \sum_{E \subseteq_0 D, G \subseteq L, j_G \in \mathcal{I}_G^*} \theta(i_E, j_G) \right) \\
 &= \left(\exp \sum_{E \subseteq_0 D} \theta(i_E) \right) \left(1 + \sum_{L \subseteq C_l^c} \exp \left(\sum_{E \subseteq_0 D, G \subseteq L, j_G \in \mathcal{I}_G^*} \theta(i_E, j_G) \right) \right) \\
 \log p^{C_l}(i_D) &= \sum_{E \subseteq_0 D} \theta(i_E) + \log \left(1 + \sum_{L \subseteq C_l^c} \exp \left(\sum_{E \subseteq_0 D, G \subseteq L, j_G \in \mathcal{I}_G^*} \theta(i_E, j_G) \right) \right) \\
 \sum_{E \subseteq_0 D} \theta(i_E) &= \log p^{C_l}(i_D) - \log \left(1 + \sum_{L \subseteq C_l^c} \exp \left(\sum_{E \subseteq_0 D, G \subseteq L, j_G \in \mathcal{I}_G^*} \theta(i_E, j_G) \right) \right)
 \end{aligned}$$

This last equality is of the form

$$\sum_{E \subseteq_0 D} \psi(E) = \phi(D) \quad (\text{A.1})$$

and therefore by Moebius inversion formula, we have

$$\psi(D) = \sum_{F \subseteq_0 D} (-1)^{|D \setminus F|} \phi(F). \quad (\text{A.2})$$

For $l = 2, \dots, k$, let $C_{<l} = H_{l-1} \setminus C_l$. Then (A.2) can be written as

$$\begin{aligned}
\theta(i_D) &= \sum_{F \subseteq_0 D} (-1)^{|D \setminus F|} \log p^{C_l}(i_F) - \sum_{F \subseteq_0 D} (-1)^{|D \setminus F|} \log \left(1 + \sum_{L \subseteq C_l^c} \exp \left(\sum_{H \subseteq_0 F, G \subseteq L, j_G \in \mathcal{I}_G^*} \theta(i_H, j_G) \right) \right) \\
&= \theta^{C_l}(i_D) - \sum_{F \subseteq_0 D} (-1)^{|D \setminus F|} \log \left(1 + \sum_{L \subseteq C_l^c} \exp \left(\sum_{H \subseteq_0 F, G \subseteq L, j_G \in \mathcal{I}_G^*} \theta(i_H, j_G) \right) \right) \\
&= \theta^{C_l}(i_D) \\
&\quad - \sum_{F \subseteq_0 D} (-1)^{|D \setminus F|} \log \left(1 + \left(\sum_{L \subseteq C_{<l}} + \sum_{L \subseteq C_{>l}} + \sum_{L \subseteq C_{<l} \cup C_{>l}} \right) \exp \left(\sum_{\substack{H \subseteq_0 F, \\ G \subseteq L, \\ j_G \in \mathcal{I}_G^*}} \theta(i_H, j_G) \right) \right) \\
&= \theta^{C_l}(i_D) \\
&\quad - \sum_{F \subseteq_0 D} (-1)^{|D \setminus F|} \log \left(1 + \sum_{L \subseteq C_{<l}} \exp \left(\sum_{\substack{H \subseteq_0 F, \\ G \subseteq L, \\ j_G \in \mathcal{I}_G^*}} \theta(i_H, j_G) \right) \right) \left(1 + \sum_{L \subseteq C_{>l}} \exp \left(\sum_{\substack{H \subseteq_0 F, \\ G \subseteq L, \\ j_G \in \mathcal{I}_G^*}} \theta(i_H, j_G) \right) \right) \\
&= \theta^{C_l}(i_D) - \sum_{F \subseteq_0 D} (-1)^{|D \setminus F|} \log \left(1 + \sum_{L \subseteq C_{<l}} \exp \left(\sum_{\substack{H \subseteq_0 F, \\ G \subseteq L, \\ j_G \in \mathcal{I}_G^*}} \theta(i_H, j_G) \right) \right) \\
&\quad - \sum_{F \subseteq_0 D} (-1)^{|D \setminus F|} \log \left(1 + \sum_{L \subseteq C_{>l}} \exp \left(\sum_{\substack{H \subseteq_0 F, \\ G \subseteq L, \\ j_G \in \mathcal{I}_G^*}} \theta(i_H, j_G) \right) \right)
\end{aligned}$$

We now want to show that the term

$$\sum_{F \subseteq_0 D} (-1)^{|D \setminus F|} \log \left(1 + \sum_{L \subseteq C_{<l}} \exp \left(\sum_{\substack{H \subseteq_0 F, \\ G \subseteq L, \\ j_G \in \mathcal{I}_G^*}} \theta(i_H, j_G) \right) \right) \quad (\text{A.3})$$

in the equation above is equal to zero.

Let F be an arbitrary subset of D and let $I = F \cap H_{l-1}$. Since $G \subseteq L \subseteq C_{<l}$, in order for $\theta(i_H, j_G)$, $H \subseteq_0 F, j_G \in \mathcal{I}_G^*$ to be non zero, it is necessary that $H \subseteq_0 I$ and therefore

$$\left(1 + \sum_{L \subseteq C_{<l}} \exp \left(\sum_{\substack{H \subseteq_0 F, \\ G \subseteq L, \\ j_G \in \mathcal{I}_G^*}} \theta(i_H, j_G) \right) \right) = \left(1 + \sum_{L \subseteq C_{<l}} \exp \left(\sum_{\substack{H \subseteq_0 I, \\ G \subseteq L, \\ j_G \in \mathcal{I}_G^*}} \theta(i_H, j_G) \right) \right). \quad (\text{A.4})$$

We see that the right hand side of (A.4) above is the same for all $F \subseteq_0 D$ that have the same intersection I with H_{l-1} . We therefore consider all such F 's. Since $D \cap R_l \neq \emptyset$, there are as many such F 's with $|D \setminus F|$ odd as there are with $|D \setminus F|$

even and therefore from (A.4), it follows that, for a given I ,

$$\sum_{\substack{F \subseteq_0 D \\ F \cap H_{l-1} = I}} (-1)^{|D \setminus F|} \log \left(1 + \sum_{L \subseteq C_{<l}} \exp \left(\sum_{\substack{H \subseteq_0 F, \\ G \subseteq L, \\ j_G \in \mathcal{I}_G^*}} \theta(i_H, j_G) \right) \right) = 0. \quad (\text{A.5})$$

Since this is true for all $I \subseteq_0 D \cap H_{l-1}$, it follows immediately from (A.5) that (A.3) is equal to zero and we have

$$\theta(i_D) = \theta^{C_l}(i_D) - \sum_{F \subseteq D} (-1)^{|D \setminus F|} \log \left(1 + \sum_{L \subseteq C_{\{>l\}}} \exp \left(\sum_{\substack{H \subseteq_0 F, \\ G \subseteq L, \\ j_G \in \mathcal{I}_G^*}} \theta(i_H, j_G) \right) \right)$$

Since $D \subseteq S_j \cap C_l$ for some $m \in \{>l\}$ and $G \subseteq L \subseteq C_m \setminus C_l$ is non empty, in the right hand side of the equation above, we have that either $\theta(i_H, j_G) = \theta^{C_m}(i_H, j_G)$ or that $\theta(i_H, j_G)$ can be expressed using (2.37) recursively and therefore $\theta(i_D)$ can be expressed in terms of $\theta^{C_m}(i_E), m \in \{>l\}, E \subseteq C_m, i_E \in \mathcal{I}_E^*$. Formula (2.37) is thus proved.